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Discontinuous solutions to the Euler equations for a van der Waals fluid

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Abstract

In this work we study the discontinuous solutions to the Euler equations for a van der Waals fluid, which contain one shock and one phase transition. We consider the general case when there is a characteristic between the shock front and the phase boundary. We establish the local existence of such solutions.

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1. Introduction

In this work, we study the discontinuous solutions to the Euler equations for a van der Waals fluid in multidimensional space variables, which contains one shock and one phase transition. There is a rich literature devoted to the discontinuous solution to the conservation laws in multidimensional space. For multidimensional shocks, one can refer to Majda [1,2], Métivier [3], Bui and Li [4], and the references therein. For multidimensional phase boundaries, one can refer to Benzoni-Gavage [5,6], Wang and Xin [7], and the references therein. In [8], Zhang and Wang proved the existence of discontinuous solutions containing one shock and one phase transition in the case when there is no characteristic between the shock front and the phase boundary. Here we shall prove the general case, namely the case when there is a characteristic between the shock front and the phase boundary.

2. Problems

For simplicity, we only study the problems in two space variables. Consider the following Euler equations for a van der Waals fluid:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) + \partial_y(\rho u v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(\rho v^2 + p(\rho)) = 0 \end{cases} \tag{2.1}$$

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where ρ is the density, (u, v) is the velocity. The pressure law $P(\tau) \equiv p(1/\tau)$ with $\tau \equiv \rho^{-1}$ being the specific volume is given by

$$P(\tau) = \frac{RT}{\tau - b} - \frac{a}{\tau^2} \quad (\tau > b) \tag{2.2}$$

where T is the temperature assumed to be a constant, R is the perfect gas constant, a and b are positive constants.

Define $U = (\rho, u, v)^{\mathrm{T}}$,

$$F_0(U) = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}, \quad F_1(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \\ \rho u v \end{pmatrix}, \quad F_2(U) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p(\rho) \end{pmatrix}$$

and $A_1(U) = (F_0'(U))^{-1} F_1'(U)$, $A_2(U) = (F_0'(U))^{-1} F_2'(U)$. Given suitable initial data $U|_{t=0} = U_{\pm}^0(x, y)$ for $\pm(x - \phi_0(y)) > 0$ with $U_{\pm}^0 \in C^1\{\pm(x - \phi_0(y)) > 0\}$ and $\{\phi_0 \in C^2\}$, we shall establish the local existence of a piecewise solution

$$U(t, x, y) = \begin{cases} U_1(t, x, y) & x < \phi(t, y) \\ U_2(t, x, y) & \phi(t, y) < x < \psi(t, y) \\ U_3(t, x, y) & x > \psi(t, y) \end{cases}$$

where $\phi(0, y) = \psi(0, y) = \phi_0(y)$, U_i $(i = 1, 2, 3)$ are C^1 in their respective domains, $\phi \in C^2$ is the phase boundary, $\psi \in C^2$ is the shock front. Denote by $\Omega_{\pm}^0 = \{\pm(x - \phi_0) > 0, t = 0\}$, $G_1 = \{x < \phi\}$, $G_2 = \{\phi < x < \psi\}$, $G_3 = \{x > \psi\}$, $\Gamma_0 = \{x = \phi_0, t = 0\}$, $\Gamma_1 = \{x = \phi\}$, and $\Gamma_2 = \{x = \psi\}$. Then U should satisfy the following free boundary problem:

$$\begin{cases} \partial_t U_i + A_1(U_i)\partial_x U_i + A_2(U_i)\partial_y U_i = 0 & \text{in } G_i \ (i = 1, 2, 3) \\ \phi_t[F_0(U)]_1 - [F_1(U)]_1 + \phi_y[F_2(U)]_1 = 0 & \text{on } \Gamma_1 \\ \left[e'(\rho) + \dfrac{(u - \phi_y v - \phi_t)^2}{2(1 + \phi_y^2)} \right]_1 + -\gamma a(j, \gamma) = 0 & \text{on } \Gamma_1 \\ \psi_t[F_0(U)]_2 - [F_1(U)]_2 + \psi_y[F_2(U)]_2 = 0 & \text{on } \Gamma_2 \\ U|_{t=0} = U_{\pm}^0 & \text{in } \Omega_{\pm}^0 \end{cases} \tag{2.3}$$

where $[\cdot]_i$ denotes the jump of a function on Γ_i $(i = 1, 2)$. $e(\rho) = \rho E(\rho)$ is the free energy per unit volume with $E(\rho)$ being the specific free energy such that $d_\rho E(\rho) = p(\rho)/\rho^2$. $j = \rho_i(u_i - \phi_y v_i - \phi_t)/\sqrt{1 + \phi_y^2}$ $(i = 1, 2)$ is the mass transfer flux across the phase boundary, which is assumed to be non-zero. And $a(j, \gamma) = j \int_{-\infty}^{+\infty} \tau'^2(\xi; j, \gamma) \mathrm{d}\xi$, where $\tau(\xi; j, \gamma)$ is the viscosity–capillarity profile satisfying the following ODE:

$$\begin{cases} \tau'' = \gamma j \tau' + \pi - p(\tau^{-1}) - j^2 \tau \\ \lim\limits_{\xi \to -\infty} \tau = \left.\dfrac{1}{\rho_1}\right|_{x=\phi}, \quad \lim\limits_{\xi \to +\infty} \tau = \left.\dfrac{1}{\rho_2}\right|_{x=\phi} \end{cases}$$

with τ', τ'' being the first and second order derivatives of τ with respect to ξ, $\pi = p(\rho_i) + j^2/\rho_i$ $(i = 1, 2)$ valued at $x = \phi$. Here the phase transition is subsonic which implies that the Mach numbers satisfy

$$M_i = \frac{1}{c_i} \left| \frac{u_i - \phi_y v_i - \phi_t}{1 + \phi_y^2} \right| < 1, \quad (i = 1, 2)$$

where $c_i = (p'(\rho_i))^{1/2}$ $(i = 1, 2)$ is the sound speed. Define $\gamma_i \cdot$ as the trace operators on Γ_i $(i = 1, 2)$ respectively. For simplicity, we denote (2.3) as

$$\begin{cases} L(U_i)U_i = 0 & \text{in } G_i \ (i = 1, 2, 3) \\ G_1(\gamma_1 U_1, \gamma_1 U_2, \phi_t, \phi_y) = 0 & \text{on } \Gamma_1 \\ G_2(\gamma_2 U_2, \gamma_2 U_3, \psi_t, \psi_y) = 0 & \text{on } \Gamma_2 \\ U|_{t=0} = U_{\pm}^0 & \text{in } \Omega_{\pm}^0. \end{cases}$$

3. Assumptions and main results

In this section, we shall give several assumptions and the main result. First we propose several assumptions.

A.1 There are two scalar functions σ_1, σ_2 and a state function U_m^0 defined on Γ_0 satisfying

$$\begin{cases} \sigma_1(F_0(U_m^0) - F_0(U_-^0)) - (F_1(U_m^0) - F_1(U_-^0)) + \phi_0'(F_2(U_m^0) - F_2(U_-^0)) = 0 \\ (e'(\rho_m^0) - e'(\rho_-^0)) + \left(\dfrac{(u_m^0 - \phi_0 v_m^0 - \sigma_1)}{1+\phi_0^2} - \dfrac{(u_-^0 - \phi_0 v_-^0 - \sigma_1)}{1+\phi_0^2} \right) + \gamma a(j_0, \gamma) = 0 \\ \sigma_2(F_0(U_+^0) - F_0(U_m^0)) - (F_1(U_+^0) - F_1(U_m^0)) + \phi_0'(F_2(U_+^0) - F_2(U_m^0)) = 0 \end{cases}$$

where $j_0 = \rho_m^0(u_m^0 - \phi_0' v_m^0 - \sigma_1)/\sqrt{1+\phi_0'^2}|_{x=\phi_0}$, $a(j_0, \gamma) = j_0 \int_{-\infty}^{+\infty} \tau_0'^2(\xi; j_0, \gamma) \mathrm{d}\xi$ and $\tau_0(\xi; j, \gamma)$ satisfies

$$\begin{cases} \tau_0'' = \gamma j_0 \tau_0' - \pi_0 - p(\tau_0^{-1}) - j_0^2 \tau_0 \\ \lim\limits_{\xi \to -\infty} = \left.\dfrac{1}{\rho_-^0}\right|_{x=\phi_0}, \quad \lim\limits_{\xi \to +\infty} = \left.\dfrac{1}{\rho_m^0}\right|_{x=\phi_0} \end{cases}$$

with $\pi_0 = p(\rho_m^0) + j_0^2/u_m^0|_{x=\phi_0}$. Denote by $\lambda_b^1 < \lambda_b^2 < \lambda_b^3$ the eigenvalues of $A_1(U_b^0) - \phi_0' A_2(U_b^0)$ for $b \in \{+, -, m\}$ respectively. We assume the following conditions:

$$\text{subsonic conditions:} \quad \lambda_-^1 < \sigma_1 < \lambda_-^2, \quad \lambda_m^1 < \sigma_1 < \lambda_m^2 \tag{3.1}$$

$$\text{Lax entropy conditions:} \quad \lambda_m^2 < \sigma_2 < \lambda_m^3, \quad \sigma_2 > \lambda_+^3 \tag{3.2}$$

are satisfied.

Remark 3.1. From (3.1) and (3.2), we see that

$$\sigma_1 < \lambda_m^2 < \sigma_2$$

which causes the major difference from the case when there is no characteristic between the shock front and phase boundary. More precisely, the difference lies in the proof of the linear estimate for the problem with one space variable which is essential to the construction of the approximate solution. In the case considered in this work, the characteristic curve of λ_m^2 may start from the shock front or the phase boundary. To establish the estimate, we should consider both situations. By integration along the characteristic the estimate could be proved.

A.2 For $s \geq 9$, the compatibility conditions up to order $s-1$ are satisfied.

A.3 For fixed $(\underline{x}, \underline{y}) \in \Gamma_0$ the planar phase transition and the planar shock front

$$U(t,x,y) = \begin{cases} U_-^0(\underline{x}, \underline{y}) & x < \sigma_1(\underline{y})t \\ U_m^0(\underline{x}, \underline{y}) & x > \sigma_1(\underline{y})t \end{cases}, \quad U(t,x,y) = \begin{cases} U_m^0(\underline{x}, \underline{y}) & x < \sigma_2(\underline{y})t \\ U_+^0(\underline{x}, \underline{y}) & x > \sigma_2(\underline{y})t \end{cases}$$

are uniformly stable.

A.4 The stability on the edge of the dihedral is given.

Remark 3.2. The detail of **A.2** and **A.4** can be found in [8] or refer to [7,3]. With **A.2** and **A.4**, we can construct an approximate solution (U^a, ϕ^a, ψ^a) that satisfies

$$\begin{cases} L(U_i^a)U_i^a = O(t^{-\lambda}) & \text{in } G_i \ (i = 1,2,3) \\ G_1(\gamma_1 U_1^a, \gamma_1 U_2^a, \phi_t^a, \phi_y^a) = O(t^{-(\lambda+1)}) & \text{on } \Gamma_1 \\ G_2(\gamma_2 U_2^a, \gamma_2 U_3^a, \psi_t^a, \psi_y^a) = O(t^{-(\lambda+1)}) & \text{on } \Gamma_2 \\ U^a|_{t=0} = U_\pm^0 & \text{in } \Omega_\pm^0 \\ \phi^a(0,y) = \psi^a(0,y) = \phi_0(y) \end{cases} \tag{3.3}$$

for a fixed large λ.

The main result of this work is as follows:

Theorem 3.1. *For any fixed $s \geq 9$, suppose $U^0_\pm \in H^s(\Omega^0_\pm)$, $U^0_m \in H^{s-1/2}(\Gamma_0)$, $\phi_0, \sigma_1, \sigma_2 \in H^{s+1/2}(R)$ and all the assumptions are satisfied; then there exists a solution (2.3) locally in time.*

4. The sketch of the proof

In this section we shall show the major steps of the proof of the theorem.

Step 1. *Linearized problem*

Following [4], we make a change of variables as follows to map the free boundary problem to a fixed boundary one:

$$\tilde{x} = \begin{cases} x - \phi & \text{in } G_1 \\ t(x-\phi)/(\psi-\phi) & \text{in } G_2, \\ x - \psi + t & \text{in} G_3 \end{cases} \quad \tilde{y} = y, \quad \tilde{t} = t, \quad \tilde{U}(\tilde{t}, \tilde{x}, \tilde{y}) = U(t, x, y).$$

With the above change of variables, we have that G_i $(i = 1, 2, 3)$ and $\Omega^0_\pm$ become $\tilde{G}_1 = \{\tilde{x} < 0\}$, $\tilde{G}_2 = \{0 < \tilde{x} < \tilde{t}\}$, $\tilde{G}_3 = \{\tilde{x} > \tilde{t}\}$, $\tilde{\Omega}^0_\pm = \{\pm\tilde{x} > 0, \tilde{t} = 0\}$ respectively. Γ_i $(i = 0, 1, 2)$ become $\tilde{\Gamma}_0 = \{\tilde{x} = 0, \tilde{t} = 0\}$, $\tilde{\Gamma}_1 = \{\tilde{x} = 0\}$, $\tilde{\Gamma}_2 = \{\tilde{x} = \tilde{t}\}$ respectively. Then the problem becomes

$$\begin{cases} \partial_t U_i + B_i \partial_x U_i + C_i \partial_y U_i = 0 & \text{in } G_i \ (i = 1, 2, 3) \\ \phi_t [F_0(U)]_1 - [F_1(U)]_1 + \phi_y [F_2(U)]_1 = 0 & \text{on } \Gamma_1 \\ \left[e'(\rho) + \dfrac{(u - \phi_y v - \phi_t)^2}{2(1+\phi_y^2)} \right]_1 + -\gamma a(j, \gamma) = 0 & \text{on } \Gamma_1 \\ \psi_t [F_0(U)]_2 - [F_1(U)]_2 + \psi_y [F_2(U)]_2 = 0 & \text{on } \Gamma_2 \\ U|_{t=0} = U^0_\pm & \text{in } \Omega^0_\pm \\ \phi(0, y) = \psi(0, y) = \phi_0(y) \end{cases} \tag{4.1}$$

where we have dropped the tildes for simplicity and $B_i = \frac{\partial \tilde{x}}{\partial t} I + \frac{\partial \tilde{x}}{\partial x} A_1(U_i) + \frac{\partial \tilde{x}}{\partial y} A_2(U_i)$, $C_i = A_2(U_i)$, With a little abuse of the notation, we denote (4.1) as

$$\begin{cases} L(\phi, \psi, U_i) U_i = 0, & \text{in } G_i \ (i = 1, 2, 3) \\ G_1(\gamma_1 U_1, \gamma_2 U_2, \phi_t, \phi_y) = 0, & \text{on } \Gamma_1 \\ G_2(\gamma_2 U_2, \gamma_3 U_3, \psi_t, \psi_y) = 0, & \text{on } \Gamma_2 \\ U|_{t=0} = U^0_\pm & \text{in } \Omega^0_\pm \\ \phi(0, y) = \psi(0, y) = \phi_0(y). \end{cases} \tag{4.2}$$

Remark. Obviously, the existence of the solution to (2.3) is equivalent to the existence of the solution to (4.2). In the remainder of this work, we shall only consider the existence of the solution to (4.2).

If we give (U, ϕ, ψ) a perturbation (V, Φ, Ψ), we get the following linearized problem:

$$\begin{cases} L(\phi, \psi, U_i) V_i = f_i & \text{in } G_i \ (i = 1, 2, 3) \\ F_1(\gamma_1 V_1, \gamma_1 V_2, \Phi_t, \Phi_y) = b_1 \Phi_t + c_1 \Phi_y + m_1 \gamma_1 V_1 + n_1 \gamma_1 V_2 = g_1 & \text{on } \Gamma_1 \\ F_2(\gamma_2 V_2, \gamma_2 V_3, \Psi_t, \Psi_y) = b_2 \Psi_t + c_2 \Psi_y + m_2 \gamma_2 V_2 + n_2 \gamma_2 V_3 = g_2 & \text{on } \Gamma_2 \\ (V, \Phi, \Psi)|_{t<0} \text{ vanish} \end{cases} \tag{4.3}$$

where the coefficients in the boundary conditions depend on U, ϕ and ψ. The detailed expressions of the coefficients in the boundary conditions can be found in [7,8].

Step 2. *Estimates for the linearized problem*

Now we establish the estimates for the linearized problem (4.3). Define $G^{\mathrm{T}}_i = G_i \cap \{0 \leq t \leq T\}$ $(i = 1, 2, 3)$ and $\Gamma^{\mathrm{T}}_i = \Gamma_i \cap \{0 \leq t \leq T\}$. We introduce the weighted Sobolev spaces in the domains G^{T}_i $(i = 1, 2, 3)$ and corresponding norms

$$H^k_\lambda(G^{\mathrm{T}}_i) = \{u \mid t^{-(\lambda-l)}\partial^\alpha_{x,y}\partial^l_t u \in L^2(G^{\mathrm{T}}_i)\ |\alpha| + l \le k\}, \quad (i = 1, 3)$$

$$|u|_{k,\lambda,T} = \left\{\sum_{|\alpha|+l\le k}(1+|\lambda|)^{2(k-l)}\int_{G^{\mathrm{T}}_i}|\partial^\alpha_{x,y}\partial^l_t u|^2 t^{-2(\lambda+l)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right\}^{1/2}, \quad (i = 1, 3)$$

$$H^k_\lambda(G^{\mathrm{T}}_2) = \left\{u \mid t^{-(\lambda-l-\alpha_x)}\partial^\alpha_{x,y}\partial^l_t u \in L^2(G^{\mathrm{T}}_2)\ |\alpha| + l \le k\right\},$$

$$|u|_{k,\lambda,T} = \left\{\sum_{|\alpha|+l\le k}(1+|\lambda|)^{2(k-\alpha_x-l)}\int_{G^{\mathrm{T}}_2}|\partial^\alpha_{x,y}\partial^l_t u|^2 t^{-2(\lambda-l-\alpha_x)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right\}^{1/2}.$$

Similarly, we define the weighted Sobolev space on the boundaries and the corresponding norms

$$H^k_\lambda(\Gamma^{\mathrm{T}}_i) = \{u \mid t^{-(\lambda-l)}\partial^s_y\partial^l_t u \in L^2(G^{\mathrm{T}}_i)\ s + l \le k\}, \quad (i = 1, 2)$$

$$\langle u\rangle_{k,\lambda,T} = \left\{\sum_{s+l\le k}(1+|\lambda|)^{2(k-l)}\int_{\Gamma^{\mathrm{T}}_i}|\partial^s_y\partial^l_t u|^2 t^{-2(\lambda-l)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right\}^{1/2} \quad (i = 1, 2).$$

Without causing any confusion, we denote by $|\cdot|_{k,T}$, $\langle\cdot\rangle_{k,T}$ the case of normal Sobolev spaces and by $|\cdot|_{k,\lambda}$, $\langle\cdot\rangle_{k,\lambda}$ the case when $T = +\infty$. Denote by $f = (f_1, f_2, f_3)$, $g = (g_1, g_2)$,

$$\begin{aligned}|||(U, \Phi, \Psi)|||^2_{k,\lambda,T} &= \lambda\left(\sum_{j=1,3}|V_j|^2_{k,\lambda,T} + |V_2|^2_{k,\lambda+1/2,T}\right) + \sum_{j=1,2}\langle\gamma_1 V_j\rangle^2_{k,\lambda,T}\\ &\quad + \sum_{j=2,3}\langle\gamma_2 V_j\rangle^2_{k,\lambda,T} + \langle\Phi\rangle^2_{k+1,\lambda+1,T} + \langle\Psi\rangle^2_{k+1,\lambda+1,T},\end{aligned}$$

$$\|f\|^2_{k,\lambda,T} = \sum_{j=1,3}|f_j|^2_{k,\lambda,T} + |f_2|^2_{k,\lambda+1/2,T}, \quad \langle g\rangle^2_{k,\lambda,T} = \sum_{j=1,2}\langle g_j\rangle^2_{k,\lambda,T}.$$

We have the following proposition:

Proposition 4.1. *Suppose $s \ge 9$ and* **A.1–A.4** *are satisfied. There exist $T_0 > 0$, $\epsilon_0 > 0$, $\lambda_0 > 0$ such that if $f_i \in H^0_{\lambda-1}(G^{\mathrm{T}}_i)$ $(i = 1, 3)$, $f_2 \in H^0_{\lambda-1/2}(G^{\mathrm{T}}_2)$, $g_i \in H^0_\lambda([0, T])$ $(i = 1, 2)$ satisfying $\|f\|^2_{0,\lambda-1} + \langle g\rangle^2_{0,\lambda}$ is finite and f, g vanish for $t < 0$, $t \ge T_0$, there is a strong solution (V, Φ, Ψ) to the problem (4.3) satisfying*

$$|||(V, \Phi, \Psi)|||^2_{0,\lambda,T} \le C\left(\lambda^{-1}\|f\|^2_{0,\lambda-1,T} + \langle g\rangle^2_{0,\lambda,T}\right) \quad \textit{for } 0 < T \le T_0, \tag{4.4}$$

where C depends on ϵ_0 and H^s norms of the coefficients of the equations and boundary conditions in (4.3). If, in addition, $\|f\|^2_{s,\lambda-1} + \langle g\rangle^2_{s,\lambda}$ is finite for $s \ge 9$ and $\partial^j_t f|_{t=0} = 0$, $\partial^j_t g|_{t=0} = 0$ for $0 \le j \le s-1$, we have

$$|||(V, \Phi, \Psi)|||^2_{s,\lambda,T} \le C\left(\lambda^{-1}\|f\|^2_{s,\lambda-1,T} + \langle g\rangle^2_{s,\lambda,T}\right) \quad \textit{for } 0 < T \le T_0. \tag{4.5}$$

To establish the estimates, we need to make another change of variables to the problem (4.3) as follows:

$$\tau = \log t, \quad X = \begin{cases} x & x < 0\\ x/t & 0 \le x \le t,\\ x - t + 1 & x \ge t\end{cases} \quad Y = y, \quad \tilde V(X, Y, \tau) = V(x, y, t)$$

and then G_i $(i = 1, 2, 3)$ and Γ_i $(i = 0, 1, 2)$ become

$$X_1 = \{x < 0\}, \quad X_2 = \{0 < x < 1\}, \quad X_2 = \{x > 1\}$$

and

$$S_0 = \{x = 0, t = 0\}, \quad S_1 = \{x = 0\}, \quad S_1 = \{x = 1\}$$

respectively. The problem (4.3) becomes

$$\begin{cases} \partial_\tau V_i + e^\tau (B_i - I)\partial_X V_i + e^\tau C_i \partial_Y V_i = e^\tau f_i & \text{in } X_i \ (i = 1, 3) \\ \partial_\tau V_2 + e^\tau (B_2 - XI)\partial_X V_2 + e^\tau C_2 \partial_Y V_2 = e^\tau f_2 & \text{in } X_2 \\ b_1 \Phi_\tau + e^\tau c_1 \Phi_Y + e^\tau m_1 \gamma_1 V_1 + e^\tau n_1 \gamma_1 V_2 = e^\tau g_1 & \text{on } S_1 \\ b_2 \Psi_\tau + e^\tau c_2 \Psi_Y + e^\tau m_2 \gamma_2 V_2 + e^\tau n_2 \gamma_2 V_3 = e^\tau g_2 & \text{on } S_2 \\ \lim\limits_{\tau \to -\infty} (V, \Phi, \Psi) = 0 \end{cases}$$

where we have dropped the tildes for simplicity.

By a finite partition of unit of the variable x in X_2, we have the partition of V_2, $\{V_2^n\}_{n=1}^N$, satisfying $V_2 = \sum_{n=1}^N V_2^n$. For those V_2^n satisfying $\operatorname{supp}\{V_2^n\} \cap S_1 = \phi$ and $\operatorname{supp}\{V_2^n\} \cap S_2 = \phi$, the estimate is obvious since there is no boundary condition for V_2^n.

For those V_2^n satisfying $\operatorname{supp}\{V_2^n\} \cap S_1 \neq \phi$ and $\operatorname{supp}\{V_2^n\} \cap S_2 = \phi$, we can restrict $\operatorname{supp}\{V_2^n\}$ to being sufficiently small that all the eigenvalues of $e^\tau(B_2 - XI)$ are non-zero in $\operatorname{supp}\{V_2^n\}$. Therefore we have the following problem for (V_1, V_2^n, Φ):

$$\begin{cases} \partial_\tau V_1 + e^\tau (B_1 - I)\partial_X V_1 + e^\tau C_i \partial_Y V_1 = e^\tau f_1 & \text{in } X_i \\ \partial_\tau V_2^n + e^\tau (B_2 - XI)\partial_X V_2^n + e^\tau C_2 \partial_Y V_2^n = e^\tau f_2 & \text{in } X_2 \\ b_1 \Phi_\tau + e^\tau c_1 \Phi_Y + e^\tau m_1 \gamma_1 V_1 + e^\tau n_1 \gamma_1 V_2^n = e^\tau g_1 & \text{on } S_1. \end{cases}$$

Define $X_i^{\mathrm{T}} = X_i \cap \{0 \le t \le T\}$ $(i = 1, 2, 3)$ and $S_i^{\mathrm{T}} = S_i \cap \{0 \le t \le T\}$ $(i = 1, 2)$. We introduce the following weighted Sobolev space: $\mathcal{H}_\lambda^k(X_i^{\mathrm{T}}) = \{u | e^{-\lambda t} u \in H^k(X_i^{\mathrm{T}})\}$ $(i = 1, 2, 3)$ and $\mathcal{H}_\lambda^k(S_i^{\mathrm{T}}) = \{u | e^{-\lambda t} u \in H^k(S_i^{\mathrm{T}})\}$ $(i = 1, 2)$. By using **A.3** and the symmetrizer constructed by Bui and Li [4], we can establish an estimate in the coordinate (τ, X, Y), which is equivalent to (4.4). Here we omit the detail for simplicity.

Similarly, when considering those V_2^n satisfying $\operatorname{supp}\{V_2^n\} \cap S_1 = \phi$ and $\operatorname{supp}\{V_2^n\} \cap S_2 \neq \phi$, we can restrict $\operatorname{supp}\{V_2^n\}$ to being sufficiently small that all the eigenvalues of $e^\tau(B_2 - XI)$ are non-zero in $\operatorname{supp}\{V_2^n\}$. Therefore we have the following problem for (V_2^n, V_3, Ψ):

$$\begin{cases} \partial_\tau V_3 + e^\tau (B_3 - I)\partial_X V_i + e^\tau C_3 \partial_Y V_3 = e^\tau f_3 & \text{in } X_3 \\ \partial_\tau V_2^n + e^\tau (B_2 - XI)\partial_X V_2^n + e^\tau C_2 \partial_Y V_2^n = e^\tau f_2 & \text{in } X_2 \\ b_2 \Psi_\tau + e^\tau c_2 \Psi_Y + e^\tau m_2 \gamma_2 V_2 + e^\tau n_2 \gamma_2 V_3 = e^\tau g_2 & \text{on } S_2. \end{cases}$$

We can also derive the estimate (4.4) in this case.

Step 3. *Iteration scheme*

To prove the existence of the solution to the nonlinear problem (4.2), we shall use the iteration scheme. Denote as E_T the extension operator such that, for any fixed $0 \le T \le T_0$, (V, Φ, Ψ) satisfies $|||(V, \Phi, \Psi)|||_{s,\lambda,T} \le \infty$ and $\partial_t^j V|_{t=0} = 0$, $\partial_t^j \Phi|_{t=0} = 0$, $\partial_t^j \Psi|_{t=0} = 0$ $(0 < j \le s - 1)$, the extended function $E_T(V, \Phi, \Psi)$ satisfies

$$\begin{cases} E_T(V, \Phi, \Psi) = (V, \Phi, \Psi) & \text{for } 0 < t < T \\ E_T(V, \Phi, \Psi) = 0 & \text{for } t > T_0 \\ |||E_T(V, \Phi, \Psi)|||_{k,\lambda,T}^2 \le C_s |||(V, \Phi, \Psi)|||_{k,\lambda,T}^2 & \text{for any } 0 \le k \le s \end{cases}$$

with a constant depending only on s. Let (U^a, ϕ^a, ψ^a) be the approximate solutions we constructed in Remark 3.2. We define the functions inductively as $(U^n, \phi^n, \psi^n) = (U^a, \phi^a, \psi^a) + E_{T_n}(V^n, \Phi^n, \Psi^n)$, where $(V^0, \Phi^0, \Psi^0) = (0, 0, 0)$, and (V^n, Φ^n, Ψ^n) is the unique solution for $0 < t \le T_n$ to the following problem provided $(V^{n-1}, \Phi^{n-1}, \Psi^{n-1})$ are known already for $0 < t \le T_{n-1}$:

$$\begin{cases} L(U_i^{n-1}, \phi^{n-1}, \psi^{n-1}) V_i^n = f_i^n & \text{in } G_i \ (i = 1, 2, 3) \\ F_{1,(\gamma_1 U_1^{n-1}, \gamma_1 U_2^{n-1}, \phi_t^{n-1}, \phi_y^{n-1})}(\gamma_1 V_1^n, \gamma_1 V_2^n, \Phi_t^n, \Phi_y^n) = g_1^n & \text{on } \Gamma_1 \\ F_{2,(\gamma_2 U_2^{n-1}, \gamma_2 U_3^{n-1}, \psi_t^{n-1}, \psi_y^{n-1})}(\gamma_2 V_2^n, \gamma_2 V_3^n, \Psi_t^n, \Psi_y^n) = g_2^n & \text{on } \Gamma_2 \end{cases}$$

where

$$F_{1,(\gamma_1 U_1^{n-1}, \gamma_1 U_2^{n-1}, \phi_t^{n-1}, \phi_y^{n-1})}(\gamma_1 V_1^n, \gamma_1 V_2^n, \Phi_t^n, \Phi_y^n)$$

$$\equiv \frac{\mathrm{d}}{\mathrm{d}s} G_1(\gamma_1 U_1^{n-1} + s\gamma_1 V_1^n, \gamma_1 U_2^{n-1} + s\gamma_1 V_2^n, \phi_t^{n-1} + s\,\Phi_t^n, \phi_y^{n-1} + s\,\Phi_y^n)\Big|_{s=0},$$

$$F_{2,(\gamma_2 U_2^{n-1}, \gamma_2 U_3^{n-1}, \psi_t^{n-1}, \psi_y^{n-1})}(\gamma_2 V_2^n, \gamma_2 V_3^n, \Psi_t^n, \Psi_y^n)$$
$$\equiv \frac{\mathrm{d}}{\mathrm{d}s} G_2(\gamma_2 U_2^{n-1} + s\gamma_2 V_2^n, \gamma_2 U_3^{n-1} + s\gamma_2 V_3^n, \psi_t^{n-1} + s\,\Psi_t^n, \psi_y^{n-1} + s\,\Psi_y^n)\Big|_{s=0},$$

and $f_i^n = -L(U_i^{n-1}, \phi^{n-1}, \psi^{n-1})U_i^a$ $(i = 1, 2, 3)$,

$$g_1^n = F_{1,(\gamma_1 U_1^{n-1}, \gamma_1 U_2^{n-1}, \phi_t^{n-1}, \phi_y^{n-1})}(\gamma_1 V_1^{n-1}, \gamma_1 V_2^{n-1}, \Phi_t^{n-1}, \Phi_y^{n-1}) - G_1(\gamma_1 U_1^{n-1}, \gamma_1 U_2^{n-1}, \phi_t^{n-1}, \phi_y^{n-1}),$$
$$g_2^n = F_{2,(\gamma_2 U_2^{n-1}, \gamma_2 U_3^{n-1}, \psi_t^{n-1}, \psi_y^{n-1})}(\gamma_2 V_2^{n-1}, \gamma_2 V_3^{n-1}, \Psi_t^{n-1}, \Psi_y^{n-1}) - G_2(\gamma_2 U_2^{n-1}, \gamma_2 U_3^{n-1}, \psi_t^{n-1}, \psi_y^{n-1}).$$

With Proposition 4.1, we can prove that for the ϵ_0 given in Proposition 4.1 there exists a $T^* > 0$ such that the solution sequence satisfies

$$|||(V^n, \Phi^n, \Psi^n)|||_{s,\lambda,T^*}^2 \le \epsilon_0 \quad \text{for } \forall n \in N \tag{4.6}$$

and for any $T \le T^*$,

$$|||(V^{n+2} - V^{n+1}, \Phi^{n+2} - \Phi^{n+1}, \Psi^{n+2} - \Psi^{n+1})|||_{s-1,\lambda,T}$$
$$\le CT|||(V^{n+1} - V^n, \Phi^{n+1} - \Phi^n, \Psi^{n+1} - \Psi^n)|||_{s-1,\lambda,T} \quad \text{for } \forall n \in N. \tag{4.7}$$

With (4.6) and (4.7), we can prove Theorem 3.1 is true.

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